Invariant and quasi-invariant measures for Hamiltonian PDEs

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$\boxed{\text{Chap 1:}}$ Invariant Gibbs measures: Part 1.

Sec 1.1: Introduction

Finite dim'l Hamiltonian dynamics on $\mathbb{R}^{2n}$:

$$(1)\quad \left.\begin{cases} \partial_t p_j = \dfrac{\partial H}{\partial q_j} \\ \partial_t q_j = -\dfrac{\partial H}{\partial p_j} \end{cases}\right\} = X \qquad j = 1, \cdots, n.$$

with $H(p, q) = H(p_1, \cdots, p_n, q_1, \cdots, q_n)$

① Liouville's thm: given $\partial_t x = f(x)$,

$$\frac{d}{dt} \text{vol} = \text{div} f.$$

$$\Rightarrow \text{div} X = \sum_{j=1}^{n} \left[\frac{\partial}{\partial p_j} X_j + \frac{\partial}{\partial q_j} X_{n+j} \right]$$

$$= \sum_{j=1}^{n} \left[\frac{\partial}{\partial p_j} \frac{\partial H}{\partial q_j} + \frac{\partial}{\partial q_j}\left(- \frac{\partial H}{\partial p_j}\right) \right] = 0$$

$\Rightarrow$ Lebesgue meas $dp\,dq = \prod_{j=1}^{n} dp_j\, dq_j$ is invariant under (1).

2 Hamiltonian H is conserved under (1).

$$\left(\frac{d}{dt} H(p(t), q(t)) = \frac{\partial H}{\partial p}\frac{\partial p}{\partial t} + \frac{\partial H}{\partial q}\frac{\partial q}{\partial t} = 0. \right.$$

$\Rightarrow$ Gibbs measures:

$$d\mu = d\mu_\beta = Z^{-1} e^{-\beta H(p,q)} \, dp\, dq$$

$\left(\beta = \text{inv temp} > 0 \right.$

are invariant under the dynamics (1).

Invariance: $\Phi(t) : (p(0), q(0)) \longmapsto (p(t), q(t))$.

$$\boxed{\mu(\Phi(-t)A)} = \mu(\{(p,q) \in \Phi(-t)A\})$$

$$= \mu(\{\Phi(t)(p,q) \in A\})$$

$$= Z^{-1}\int_A e^{-\beta \underbrace{H(p(t), q(t))}_{\substack{\shortparallel \\ H(p(0),\, q(0))}}} \underbrace{dp(t)\, dq(t)}_{dp(0)\, dq(0)}$$

$$\boxed{= \mu(A)}$$

<u>Rmk:</u> Suppose that F is conserved under (1)

$$\Rightarrow \quad d\mu_F = Z^{-1} e^{-F(p,q)} \, dp\, dq$$

is also invariant (for reasonable F).

Q: Why do we care about invariant meas?

① can view (1) as a dynamical system with meas-preserving transf $T := \Phi(1)$.

$$T : \big(p(0), q(0)\big) \longmapsto \big(p(t), q(t)\big)\Big|_{t=1}.$$

$\Rightarrow$ <u>Poincaré recurrence thm</u>.

(i) $\forall$ meas A with $\mu(A) > 0$,

$\exists\, N \in \mathbb{N}$ s.t. $\mu(A \cap T^{-N} A) > 0$.

(ii) a.e. $(p, q) \in \mathbb{R}^{2n}$ are stable according to Poisson. i.e.

see Zhidkov '01 Lec Notes in Math.

$\exists \{t_j\} \to \infty$ s.t.

$$\big(p(t_j), q(t_j)\big) \longrightarrow \big(p(0), q(0)\big)$$

Also, <u>Furstenberg's multiple recurrence thm</u>

$\forall\, A$ with $\mu(A) > 0$ and $k \in \mathbb{N}$,

$\exists\, N \in \mathbb{N}$ s.t.

$$\mu(A \cap T^{-N} A \cap \cdots \cap T^{-(k-1)N} A) > 0.$$

② μ is "supposed" to describe a typical long time behavior of solutions.

Sec 1.2: Hamiltonian PDEs

- Nonlinear Schrödinger equation (NLS):

$$\text{(NLS)} \quad i\partial_t u + \Delta u = \pm |u|^{p-2} u \quad \text{on } \mathbb{T}^d.$$

- (generalized) Korteweg-de Vries equation (gKdV):

$$\text{(gKdV)} \quad \partial_t u + u_{xxx} = \pm\, u^{p-2} \partial_x u \quad \text{on } \mathbb{T}.$$

- $p=3$: KdV $(u\partial_x u)$
- $p=4$: modified KdV (mKdV with $u^2 \partial_x u$)

These are Hamiltonian PDEs with

$$H(u) = \frac{1}{2}\int_{\mathbb{T}^d} |\nabla u|^2 dx \pm \frac{1}{p}\int_{\mathbb{T}^d} |u|^p dx.$$

$$(\text{NLS}) \iff \partial_t u = -i \frac{\partial H}{\partial \bar{u}}.$$

Symplectic space $L^2(\mathbb{T}^d)$

symplectic form $\omega(f,g) = \operatorname{Im} \int_{\mathbb{T}^d} f(x)\overline{g(x)}\,dx$

$$dH\big|_u(\phi) = \omega\left(\phi, -i\frac{\partial H}{\partial \bar{u}}\right)$$

Gâteaux derivative $= \frac{d}{d\varepsilon} H(u+\varepsilon\phi)\big|_{\varepsilon=0}.$

<u>On the Fourier side:</u>

$$p_n = \operatorname{Re} \hat{u}_n, \qquad q_n = \operatorname{Im} \hat{u}_n.$$

$$H(p,q) = \frac{1}{2}\sum |n|^2 (p_n^2 + q_n^2)$$

$$(\text{NLS}) \quad \partial_t u = i\Delta u + \text{nonlin}$$

$$\iff \partial_t \hat{u}_n = -i|n|^2 \hat{u}_n$$

$$\iff \partial_t \begin{pmatrix} p_n \\ q_n \end{pmatrix} = \begin{pmatrix} \frac{\partial H}{\partial q_n} \\ -\frac{\partial H}{\partial p_n} \end{pmatrix}.$$

Note that $H(u)$ is conserved under the dynamics.

Moreover, we use the conservation of mass

$$M(u) = \int |u|^2 dx.$$

Natural question:

Is the Gibbs measure

$$\text{"}d\mu = Z^{-1} e^{-\beta H(u)} du\text{"}$$

invariant under NLS (or gKdV)?

$\beta = 1.$

$$d\mu = Z^{-1} e^{-H(u)} du$$

$$= Z^{-1} e^{\mp \frac{1}{p} \int |u|^p dx} \underbrace{e^{-\frac{1}{2} \int |\nabla u|^2 dx} du}_{\text{Wiener measure on } \mathbb{T}^d.}$$

In the following, we consider

$$d\mu = Z^{-1} e^{-H(u) - \frac{1}{2} M(u)} du$$

for NLS in order to avoid the problem at freq 0.

Sec 1.3: Gaussian measures on H^s.

Consider
$$d\rho_s = Z^{-1} e^{-\frac{1}{2}\|u\|^2_{H^s}} du$$

as a limit of
$$d\rho_{s,N} = Z_N^{-1} e^{-\frac{1}{2}\|P_{\le N} u\|^2_{H^s}} d P_{\le N} u$$

$$= Z_N^{-1} \prod_{|n|\le N} \underbrace{e^{-\frac{1}{2}\langle n\rangle^{2s}|\hat{u}_n|^2} d\hat{u}_n}_{\substack{\text{Gaussian meas on } \mathbb{C}.\\ \text{mean } 0,\ \text{var} = 2\langle n\rangle^{-2s}.}}$$

· Namely, $\rho_{s,N}$ is the induced prob meas under the map

$$\omega \longmapsto u^N = \sum_{|n|\le N} \frac{g_n(\omega)}{\langle n\rangle^s} e^{in\cdot x}$$

where $\{g_n\}$ = indep. standard $\mathbb{C}$-valued Gaussian r.v.'s.

$$\mathbb{E}(g_n) = 0, \quad \mathrm{Var}(g_n) = 2.$$

<u>Problem:</u> We can <u>not</u> take a limit $N \to \infty$ in $H^s(\mathbb{T}^d)$!!

Let $\sigma \in \mathbb{R}$ and $M > N$.

$$\mathbb{E}\left[\| u^M - u^N \|^2_{H^\sigma(\mathbb{T}^d)}\right]$$

$$= \mathbb{E} \sum_{N < |n| \le M} |g_n|^2 / \langle n \rangle^{2s - 2\sigma}$$

$$\sim \sum_{N < |n| \le M} \frac{1}{\langle n \rangle^{2s-2\sigma}} \to 0$$

if and only if $2s - 2\sigma > d$

(2) $$\iff \boxed{\sigma < s - \frac{d}{2}}$$

If (2) is satisfied, then

u^N converges in $L^2(\Omega ; H^\sigma(\mathbb{T}^d))$.

and define ρ_s as the induced prob meas under

$$\omega \mapsto u = \sum_{n \in \mathbb{Z}^d} \frac{g_n}{\langle n \rangle^s} e^{in \cdot x}$$

$\Leftarrow$ prob meas on $H^\sigma(\mathbb{T}^d)$

Lec Notes in Math.
Gross '65, Kuo '75

separable

Abstract Wiener space : $H = \infty$ - dim'l Hilbert space

$$" d\rho = Z^{-1} e^{-\frac{1}{2}\|u\|_H^2} du "$$

$\Leftarrow$ NOT countably additive if $\dim H = \infty$.

densely & conti.

$\Rightarrow$ Must work on a larger space $B \supset H$

separable Banach space

We say (B, H, ρ) is an abstract Wiener space

if
$$\int_B e^{i\langle u, \varphi\rangle} d\rho(u) = e^{-\frac{1}{2}\|\varphi\|_{H^*}^2}$$

B-B^* duality.

for all $\varphi \in B^* \subset H^*$.

Rmk: ① $i : H \hookrightarrow B$

The triplet (i, H, B) is often called an abstract Wiener space.

② B = completion of H under a "measurable" seminorm $\|\cdot\|_B$:

$\forall \varepsilon > 0 \ \exists P_\varepsilon \in \mathcal{F}$ = collection of finite dim'l ortho proj.

s.t.
$$\rho(\|Pu\|_B > \varepsilon) < \varepsilon$$

for all $P \in \mathcal{F}$ orthogonal to P_ε.

ex: $H = H^s$.

$B = H^\sigma$, $\quad \sigma < s - \frac{d}{2}$.

$W^{\sigma,p}$

$\mathcal{F}L^{\sigma,p}$, $\quad (\sigma - s)p < -d$

$$\|f\|_{\mathcal{F}L^{\sigma,p}(\mathbb{T}^d)} = \|\langle n\rangle^\sigma \hat{f}_n\|_{\ell^p(\mathbb{Z}^d)}$$

Lemma 1.1 (Tail estimate). Let $\sigma < s - \frac{d}{2}$.

Then,
$$\rho_s\left(\{\|u\|_{H^\sigma} > K\}\right) \le C e^{-cK^2}$$

for all $K > 0$.

Rmk: This follows from Fernique's integrability thm:

$$\int_B e^{c\|u\|_B^2} d\rho(u) < \infty.$$

Pf: By Chebyshev's ineq,

$$e^{cK^2} \rho_s(B_K^c) \le \int_{H^\sigma} e^{c\|u\|_{H^\sigma}^2} d\rho_s(u)$$

$$= \prod_{n\in\mathbb{Z}^d} \int_{\mathbb{C}} e^{c\langle n\rangle^{2\sigma - 2s} |g_n|^2} e^{-\frac{1}{2}|g_n|^2} \frac{dg_n}{2\pi}$$

$$= \prod_{n \in \mathbb{Z}^d} \frac{1}{1 - 2c\langle n\rangle^{2\sigma - 2s}}$$

$$\left(\mathbb{E}\left[e^{aX^2}\right] = (1-2a)^{-1/2}, \quad a < 1/2 \right.$$

$X \sim N_{\mathbb{R}}(0,1)$

$$= \prod_{n \in \mathbb{Z}^d} \left(1 + \frac{2c\langle n\rangle^{2\sigma - 2s}}{1 - 2c\langle n\rangle^{2\sigma - 2s}} \right)$$

$< \infty$ iff $\sigma < s - \frac{d}{2}$

$\Big($ $a_n > 0$. Then

$\prod(1 + a_n) < \infty \iff \sum a_n < \infty$

$\boxed{/}$

Sec 1.4: Construction of Gibbs measure on $\mathbb{T}$

Fix $d = 1$:

$$\text{"} d\mu = Z^{-1} e^{-H(u) - \frac{1}{2}M(u)} du$$

$$= Z^{-1} e^{\mp \frac{1}{p}\int |u|^p dx} d\rho \text{"}$$

where

$$\rho = \rho_1 = \text{Wiener meas on } \mathbb{T}.$$

actually, o-v.

- Under ρ, we have

(3) $$u = \sum_n \frac{g_n}{\langle n \rangle} e^{inx}.$$

Bényi-Oh '11 Appendix A

(Compare this with Fourier-Wiener series of a Brownian ~~motion~~ motion on $[0, 2\pi)$:

$$B(t) = g_0 t + \sum_{n \neq 0} \frac{g_n}{\sqrt{2\pi}\, in} e^{int} - \sum_{n \neq 0} \frac{g_n}{\sqrt{2\pi}\, in}$$

- Defocusing case (with $-$ sign):

$$(3) \Rightarrow u \in H^s(\mathbb{T}) \quad s < \tfrac{1}{2} \quad \text{a.s.}$$

$$\underset{\text{Sobolev}}{\Rightarrow} u \in L^p(\mathbb{T}), \quad p < \infty \quad \text{a.s.}$$

$$\Rightarrow \quad 0 < e^{-\frac{1}{p}\int |u|^p dx} \leq 1 \quad \text{a.s.}$$

$\Rightarrow$ μ is a prob meas on $H^s(\mathbb{T}) \setminus H^{1/2}(\mathbb{T})$
$s < 1/2$.

$\mu \ll \rho$.

- <u>Focusing case</u>: Let $p > 2$.

$$\int_{\mathbb{T}} |u|^p \geq \|u\|_{L^2}^p = \left(\sum |\hat{u}_n|^2\right)^{p/2}$$

$$\geq \sum \left|\frac{g_n}{\langle n \rangle}\right|^p \qquad (\ell^2 \subset \ell^p)$$

$$\Rightarrow \mathbb{E}_\rho\left[e^{\int |u|^p}\right] \geq \prod_n \mathbb{E}\left[e^{|g_n|^p / \langle n \rangle^p}\right] = \infty.$$

$$\left(\mathbb{E}\left[e^{|g|^p}\right] = \frac{1}{2\pi}\int e^{|g|^p} e^{-|g|^2/2} dg = \infty. \right.$$

- <u>Remedy</u>: Mass is conserved.

$\Rightarrow$ Introduce a mass cutoff.

Prop 1.2: (Lebowitz-Rose-Speer '88, Bourgain '94)

(i) $p < 6$

$$R(u) = R_r(u) := \mathbb{1}_{\{\|u\|_{L^2} \leq r\}} \, e^{\frac{1}{p}\int |u|^p} \in L^q(d\rho) \tag{4}$$

$\forall\, r, q < \infty.$

(ii) $p = 6$: (4) holds if r is sufficiently small.

(In fact, if $r < \|Q\|_{L^2(\mathbb{R})}$

Q — ground state

related to (minimal mass) finite time blowup solution

(iii) If $p > 6$ or $p = 6$ and $r \geq \|Q\|_{L^2(\mathbb{R})}$, then (4) fails.

Rmk: $p = 6$: mass-critical NLS

$\Leftarrow$ lowest power for which we have finite time blowups.

$\boxed{\text{Lemma 1.3}}$: $\{\tilde{g}_n\}$ indep standard $\mathbb{R}$-valued Gaussian r.v.'s.

Then,

$$P\left[\left(\sum_{n=1}^{M} \tilde{g}_n^2\right)^{1/2} \geq R\right] \leq e^{-\frac{1}{4}R^2}, \quad R \geq 3M^{1/2}, \ M \geq 1$$

Pf: $0 < t < 1/2$

$$(\text{LHS}) \leq e^{-tR^2} \mathbb{E}\left[e^{t\sum_{n=1}^{M} \tilde{g}_n^2}\right]$$

$$= (1-2t)^{-M/2} e^{-tR^2}.$$

Choose $t = \frac{1}{2}\left(1 - \frac{M}{R^2}\right)$.

$$\Longrightarrow \quad (\text{LHS}) = \left(\frac{R^2}{M}\right)^{M/2} e^{-\frac{1}{2}R^2 + \frac{1}{2}M}$$

$$\leq e^{\frac{M}{2}\log\frac{R^2}{M} + \left(\frac{1}{18} - \frac{1}{2}\right)R^2} \leq e^{-\frac{1}{4}R^2}$$

$\left(\log x \leq \frac{x}{4} \text{ for } x \geq 9 \right)$

$\square$

Pf of Prop 1.2 (i) and (ii):

- By Bernstein's ineq,

$$\| P_{\leq M_0} u \|_{L^p} \leq c M_0^{\frac{1}{2}-\frac{1}{p}} \underbrace{\| P_{\leq M_0} u \|_{L^2}}_{\leq r}$$

$\lambda \geq 1$ Choose M_0 s.t.

$$\frac{1}{2} \lambda = c M_0^{\frac{1}{2}-\frac{1}{p}} r$$

i.e.

(5) $$M_0 \sim \left(\frac{\lambda}{r}\right)^{\frac{1}{\frac{1}{2}-\frac{1}{p}}}$$

- Let $M_j = 2^j M_0$. Then, we have

$$\| P_{M_j} u \|_{L^p} \leq c M_j^{\frac{1}{2}-\frac{1}{p}} \| P_{M_j} u \|_{L^2}.$$

- Let $\{\sigma_j\}$ s.t. $\sum_{j \geq 1} \sigma_j = \frac{1}{2}$.

$$\Leftarrow \text{ set } \sigma_j = C 2^{-\varepsilon j} = C M_0^{\varepsilon} M_j^{-\varepsilon}.$$

By dyadic pigeon hole principle,

$$\rho\left(\| u \|_{L^p} > \lambda, \ \| u \|_{L^2} \leq r \right)$$

$$\leq \sum_{j=1}^{\infty} \underbrace{\rho\left(\| P_{M_j} u \|_{L^p} > \sigma_j \lambda \right)}_{\leq \rho\left(\| P_{M_j} u \|_{L^2} > c \sigma_j M_j^{\frac{1}{p}-\frac{1}{2}} \lambda \right)}$$

$$P\left(\|P_{M_j}u\|_{L^2} > c\sigma_j M_j^{\frac{1}{p}-\frac{1}{2}}\lambda\right)$$

$$\sim P\left(\Big(\sum_{|n|\sim M_j}|g_n(\omega)|^2\Big)^{1/2} \gtrsim \underbrace{\sigma_j M_j^{\frac{1}{p}+\frac{1}{2}}\lambda}\right)$$

$$R_j := \sigma_j M_j^{\frac{1}{p}+\frac{1}{2}}\lambda \qquad \lambda \geq 1$$

$$\gtrsim M_0^{\varepsilon} M_j^{\frac{1}{2}+\frac{1}{p}-\varepsilon}$$

$$\gtrsim M_0^{\varepsilon} M_j^{\frac{1}{2}+}.$$

Lemma 1.3.

$$\Longrightarrow \quad P\left(\|u\|_{L^p} > \lambda,\ \|u\|_{L^2} \leq r\right)$$

$$\leq \sum_{j=1}^{\infty} e^{-cR_j^2}$$

$$R_j^2 \sim M_0^{\frac{2}{p}+1}\lambda^2\, 2^{\left(\frac{2}{p}+1-2\varepsilon\right)j}$$

$$\sim e^{-c\lambda^2 M_0^{\frac{2}{p}+1}}$$

$$\overset{(5)}{\sim} e^{-c\lambda^{\frac{4p}{p-2}} r^{-\frac{2p+4}{p-2}}} \lesssim \begin{cases} e^{-c\lambda^{p+\delta}}, & p > 6 \\ e^{-c\lambda^p} & \text{with } c \gg 1 \text{ if } p = 6 \end{cases}$$

$$\frac{4p}{p-2} > p \quad \text{for } p < 6$$

$\square$

$$R_r(u) = \mathbb{1}_{\{\|u\|_{L^2} \le r\}} e^{\frac{1}{p}\int |u|^p}$$

$$R_{N,r}(u) = \mathbb{1}_{\{\|P_{\le N} u\|_{L^2} \le r\}} e^{\frac{1}{p}\int |P_{\le N} u|^p}$$

$\boxed{\underline{\text{Cor 1.4:}}}$ $\forall q < \infty$,

$R_{N,r}(u) \to R_r(u)$ in $L^q(d\rho)$ as $N \to \infty$.

In particular,

$$d\mu_{N,r} = Z_N^{-1} R_{N,r}(u)\, d\rho$$

converges "uniformly" to $\mu = \mu_r$.

i.e. $\forall \varepsilon > 0$, $\exists N_0 \in \mathbb{N}$ s.t.

$$|\mu_{N,r}(A) - \mu(A)| < \varepsilon$$

$\forall N \ge N_0$ and measurable A.

Pf: $\int |P_{\leq N} u|^p \to \int |u|^p$ a.s.

$\Rightarrow R_{N,r}(u) \to R_r(u)$ a.s.

$\Rightarrow$ By Egoroff's thm,

$R_{N,r}(u) \to R_r(u)$ almost uniformly
$\Rightarrow$ in measure.

- Given $\varepsilon > 0$,

let $A_{N,\varepsilon} = \{ u \in H^{\frac{1}{2}}(\mathbb{T}) : |R_{N,r}(u) - R_r(u)| \leq \frac{1}{2}\varepsilon \}$.

Then, $\rho(A^c_{N,\varepsilon}) \to 0$.

$$\Rightarrow \| R_{N,r} - R_r \|_{L^q(\rho)}$$

$$\leq \| (R_{N,r} - R_r) \mathbb{1}_{A_{N,\varepsilon}} \|_{L^q}$$

$$+ \| (R_{N,r} - R_r) \mathbb{1}_{A^c_{N,\varepsilon}} \|_{L^q}$$

$$\leq \frac{1}{2}\varepsilon + \underbrace{\left(\| R_{N,r} \|_{L^{2q}} + \| R_r \|_{L^{2q}} \right)}_{\leq C < \infty} \left\{ \rho(A^c_{N,\varepsilon}) \right\}^{1/2p}$$

$$< \varepsilon.$$

$\square$

Sec 1.5: Invariance of the Gibbs measure.

3 scenarios.

· Case I: We have an a priori (deterministic) global well-posedness in $\mathrm{supp}(\mu) \supset H^{\frac{1}{2}-}(\mathbb{T})$.

ex: KdV, cubic NLS.

$\Rightarrow$ easy

Case II: Only a priori local-in-time dynamics.

Idea: use invariance (in place of a conservation law) to construct global-in-time dynamics

$$\begin{tikzcd} \text{invariance} \arrow[rr, bend right] & & \text{a.s. GWP} \arrow[ll, bend right] \end{tikzcd}$$

Bourgain '94: use invariance of the finite dim'l Gibbs meas associated to the finite dim'l approx

$\Rightarrow$ a.s. GWP. $\Rightarrow$ invariance.

Case III : No local well-posedness.

(III.1) probabilistic local-in-time Cauchy theory
Bourgain '96, Burq–Tzvetkov '07

(III.2) compactness argument
(for measures on space-time functions)

Burq–Thomann–Tzvetkov '14, Oh–Thomann '15.

In the following, we focus on Case II.

(NLS) $i\partial_t u + \Delta u = \pm |u|^{p-2} u.$

Suppose that we have subcritical LWP in $\mathrm{supp}(\mu)$.
($\Leftarrow$ Bourgain '93)

i.e. time of local existence $\delta \sim (1 + \|u_0\|_B)^{-\theta}$

$$B = H^{\frac{1}{2}-\varepsilon}(\mathbb{T})$$

Then, the same argument yields LWP of the following "finite dimensional" approx

$$\text{(FNLS)} \begin{cases} i\partial_t u^N + \Delta u^N = \pm P_{\leq N}(|u^N|^{p-2} u^N) \\ u^N|_{t=0} = u_0 \end{cases}$$

for $|t| \leq \delta$.

- (FNLS) is NOT finite dim'l.

  - $\partial_t u^N = -i \dfrac{\partial H_N}{\partial \overline{u^N}}$

  where $H_N(u^N) = \frac{1}{2}\int |\partial_x u^N|^2 \pm \frac{1}{p}\int |P_{\leq N} u^N|^p$

  lin. dynamics on high freq.

① $v^N := P_{\leq N} u^N$ satisfies

$$(\text{FNLS}_{low}) \quad i\partial_t v^N + \partial_x^2 v^N = \pm P_{\leq N}(|v^N|^{p-2} v^N)$$

finite dim'l system of ODEs (on the Fourier side)

② $P_{>N} u^N$ evolves linearly:

$$(\text{FNLS}_{high}) \quad \partial_t \widehat{u^N}(n) = -i n^2 \widehat{u^N}(n), \quad |n| > N.$$

- As a lin. eqn, (FNLS_{high}) is GWP.
- (FNLS_{low}) preserves the L^2-norm:

$$\|v^N\|_{L^2} = \Big(\sum_{|n|\leq N} |\widehat{v^N}(n)|^2\Big)^{1/2}$$

$$= \text{Euclidean distance on } \mathbb{C}^{2N+1}$$

$\Rightarrow$ (FNLS_{low}) is GWP.

$\Rightarrow$ (FNLS) is GWP for each $N \in \mathbb{N}$.

but there is NO uniform (in N) control on $\|u^N(t)\|_B$.

Write $\rho = \rho_N \otimes \rho_N^\perp$

$$\begin{cases} d\rho_N = Z_N^{-1} e^{-\frac{1}{2}\|P_{\le N} u\|_{H^1}^2} d P_{\le N} u \\ d\rho_N^\perp = Z_N^{-1} e^{-\frac{1}{2}\|P_{>N} u\|_{H^1}^2} d P_{>N} u \end{cases}$$

Then,

① $d\mu_{N,low} = Z_N^{-1} R_{N,r}(v^N) d\rho_N$ is invariant under ($FNLS_{low}$).

② $\rho_N^\perp$ is invariant under ($FNLS_{high}$).

$\Rightarrow$ $\mu_N := \mu_{N,low} \otimes \rho_N^\perp$ is invariant under (FNLS).

$\boxed{\text{Prop 1.5}}$ (Bourgain '94)

$\forall\ T < \infty,\ \varepsilon > 0,\ \exists\ \Omega_N = \Omega_N(T, \varepsilon)$ s.t.

(i) $\mu_N(\Omega_N^c) < \varepsilon$

(ii) For $u_0 \in \Omega_N$, $\exists!$ soln u^N to (FNLS)

s.t.

$s<\frac{1}{2}$ $$\| u^N(t) \|_{H^s} \lesssim \left(\log \frac{T}{\varepsilon}\right)^{1/2}, \quad |t| \le T.$$

Pf: Local theory:

$$\| u_0 \|_{H^s} \le K \Rightarrow \| u^N(t) \|_{H^s} \le 2K$$

for $|t| \le \delta \sim K^{-\theta}$.

Let
$$\Omega_N = \bigcap_{j=-[T/\delta]}^{[T/\delta]} \Phi_N(j\delta)\left(\|u_0\|_{H^s} \le K \right)$$

(i) By invariance of μ_N,

$$\mu_N(\Omega_N^c) \lesssim T/\delta\ \mu_N(\cancel{\Phi_N(j\delta)}(B_K^c))$$

$$\lesssim T K^\theta e^{-cK^2} < \varepsilon$$

Prop 1.2 & Lemma 1.1.

by choosing $K \sim \left(\log \frac{T}{\varepsilon}\right)^{1/2}$.

(ii) By construction,

$$\| u^N(j\delta)\|_{H^s} \le K, \quad j = 0, \pm 1, \cdots, \pm [T/\delta]$$

$\Rightarrow$ By local theory,

$$\| u^N(t)\|_{H^s} \le 2K \sim \left(\log \frac{T}{\varepsilon}\right)^{1/2}, \quad |t| \le T$$

$\square$

$\boxed{\text{Lemma 1.6}}$ (Approximation Lemma) $\quad s < 1/2$

$$u_0 \in H^s \text{ with } \|u_0\|_{H^s} \le K.$$

Suppose soln u^N to (FNLS) with $u^N|_{t=0} = u_0$ satisfies

$$\| u^N(t)\|_{H^s} \le K, \quad |t| \le T.$$

Then, $\exists !$ soln u to (NLS) on $[-T, T]$ with $u|_{t=0} = u_0$.

Moreover, $\exists\ C_0, C_1, C_2$ s.t.

$$\| u(t) - P_N u^N(t)\|_{H^{s_1}} \le C_0\, e^{C_1 (1+K)^{C_2} T} K \underbrace{N^{s_1 - s}}_{\to 0}$$

for $s_1 < s$.

<u>Pf</u>: (FNLS) and (NLS) with $u^N|_{t=0} = u|_{t=0} = u_0$ are locally well-posed on $[-\delta, \delta]$, $\delta \sim (1+K)^{-\theta}$ indep of N.

$\Rightarrow$ With $v^N = P_{\leq N} u^N$,

$$\| u - v^N \|_{X^{s_1, b}}$$

$$\lesssim \underbrace{\| \underset{\substack{\uparrow \\ \text{smooth cutoff}}}{\eta(t)} S(t) P_{>N} u_0 \|_{H^{s_1}}}_{\lesssim K N^{s_1 - s}} + \text{nonlin. terms}$$

$$① \ \underbrace{(Id - P_{\leq N})(|u|^{p-2} u)}_{\lesssim \delta^\theta N^{s_1 - s} K^{p-1}}$$

$$+ ② \ P_{\leq N} \left(|u|^{p-2} u - |v^N|^{p-2} v^N \right)$$

$$\lesssim \delta^\theta \cdot K^{p-2} \| u - v^N \|_{X^{s_1, b}}$$

$$\Longrightarrow \| u - v^N \|_{X^{s_1, b, \delta}} \leq C K N^{s_1 - s} + \tfrac{1}{2} \| u - v^N \|_{X^{s_1, b, \delta}}$$

$$\Longrightarrow \| u - v^N \|_{X^{s_1, b, \delta}} \leq 2 C K N^{s_1 - s}$$

Repeating $\sim T/\delta$ many times, we obtain

$$\| u(t) - v^N(t) \|_{H^{s_1}} \lesssim \underbrace{e^{T/\delta}}_{\longrightarrow\, e^{c_1 (1+K)^{c_2} T}} K N^{s_1 - s}$$

$\boxtimes$

Prop 1.7 (Almost a.s. GWP) $s < 1/2$

Given $T, \varepsilon > 0$, $\exists\, \Omega_{T,\varepsilon}$ s.t.

(i) $\mu(\Omega^c_{T,\varepsilon}) < \varepsilon$

(ii) For $u_0 \in \Omega_{T,\varepsilon}$, $\exists !$ soln u to (NLS) on $[-T,T]$

s.t. $\|u(t)\|_{H^s} \lesssim \left(\log \frac{T}{\varepsilon}\right)^{1/2}$, $|t| \le T$.

Pf: Let $\Omega_N(T,\varepsilon)$ be as in Prop 1.5.

From Prop 1.5, $\|\Phi_N(t)(u_0)\|_{H^s} \le 2K$

for $|t| \le T$ and $u_0 \in \Omega_N$.

- By Lemma 1.6, $\exists\, N_1 \in \mathbb{N}$ s.t.

$$\|u(t) - u^N(t)\|_{H^{s_1}} \ll 1, \quad |t| \le T$$

for $N \ge N_1$

$\Rightarrow \|u(t)\|_{H^{s_1}} \lesssim K \sim \left(\log \frac{T}{\varepsilon}\right)^{1/2}$, $|t| \le T$.

- Also, $\mu(\Omega_N^c) \overset{\text{C.-S.}}{\lesssim} \Big(\int_{\Omega_N^c} \underbrace{\mathbb{1}_{\{\|u\|_{L^2} \le r\}}}_{\le \mathbb{1}_{\{\|P_{\le N} u\|_{L^2} \le r\}}} d\rho \Big)^{1/2}$

$\overset{\text{C.-S.}}{\lesssim} \left(\mu_N(\Omega_N^c)\right)^{\frac{1}{4}} < \varepsilon.$

$\boxtimes$

Thm 1.8: NLS is a.s. GWP with respect to μ.

Pf: Given $\varepsilon > 0$, let $T_j = 2^j$, $\varepsilon_j = \frac{\varepsilon}{2^j}$

$\Rightarrow$ Construct $\Omega_j = \Omega_{T_j, \varepsilon_j}$.

· Let $\Omega_\varepsilon = \bigcap_{j=1}^{\infty} \Omega_j$

(i) $\mu(\Omega_\varepsilon^c) < \sum_{j=1}^{\infty} \frac{\varepsilon}{2^j} = \varepsilon$

(ii) If $u_0 \in \Omega_\varepsilon$, then $\exists!$ soln u on $[-T_j, T_j]$

for any $j \in \mathbb{N} \Rightarrow$ global soln.

· Let $\Sigma = \bigcup_{\varepsilon > 0} \Omega_\varepsilon$

(i) $\mu(\Sigma^c) = \inf_{\varepsilon > 0} \varepsilon = 0$.

(ii) If $u_0 \in \Sigma$, then $u_0 \in \Omega_\varepsilon$ for some $\varepsilon > 0$.

$\Rightarrow \exists!$ global soln u with $u|_{t=0} = u_0$

$\square$

Thm 1.9: The Gibbs measure μ is invariant under NLS.

Pf: By time reversibility of $\Phi(t)$, it suffices to show

$$\mu(A) \leq \mu(\Phi(t) A) \tag{6}$$

for all measurable set $A \subset \Sigma$ ant $t \in \mathbb{R}$

By inner regularity, we have

$$\mu(A) = \sup_{\substack{F \subset A \\ \text{closed in } H^s.\ s<1/2}} \mu(F)$$

i.e. $\exists \{F_n\}$ closed in H^s s.t. $F_n \subset A$ and

$$\mu(A) = \lim_{n\to\infty} \mu(F_n)$$

- Suffices to prove (6) for closed sets.

Then,
$$\begin{aligned} \mu(A) &= \lim_{n\to\infty} \mu(F_n) \\ &\le \overline{\lim_{n\to\infty}}\, \mu(\Phi(t)F_n) \\ &\le \mu(\Phi(t)A) \end{aligned}$$

$F_n \subset A$.

- Given a closed set $F \subset H^s$,

let $K_n = \{u \in F : \|u\|_{H^\sigma} \le n\}$, $s<\sigma<1/2$.

Then, K_n is compact in H^s.

$\Rightarrow$ Suffices to prove (6) for compact sets.

Then,
$$\begin{aligned} \mu(F) &= \lim_{n\to\infty} \mu(K_n) \\ &\le \overline{\lim_{n\to\infty}}\, \mu(\Phi(t)K_n) \\ &\le \mu(\Phi(t)F) \end{aligned}$$

Lemma 1.1 & Prop 1.2

$K_n \subset F$

Assume bdd. in H^{σ}

· Let K be a compact set in H^s.

· Since $\mu_N \rightharpoonup \mu$ & Portmanteau thm, we have

(7) $$\mu\left(\Phi(t)K + \overline{B_\varepsilon}\right) \geq \overline{\lim}\, \mu_N\left(\Phi(t)K + \overline{B_\varepsilon}\right)$$

· Fix $t \ll 1$. Then,

local theory

$$\Phi_N(t)\left(K + B_\delta\right) \subset \Phi_N(t)K + B_{\varepsilon/2}$$

$$\subset \Phi(t)K + B_\varepsilon$$

Approximation lemma (Lemma 1.6).

$\Rightarrow$ By invariance of μ_N,

(8) $$\mu_N(K + B_\delta) = \mu_N\left(\Phi_N(t)(K + B_\delta)\right)$$

$$\leq \mu_N\left(\Phi(t)K + B_\varepsilon\right)$$

Hence,

$$\mu(K) \leq \mu(K + B_\delta)$$

$$\leq \underline{\lim}\, \mu_N(K + B_\delta)$$

$$\underset{(8)}{\leq} \underline{\lim}\, \mu_N\left(\Phi(t)K + B_\varepsilon\right)$$

$$\leq \overline{\lim}\, \mu_N\left(\Phi(t)K + \overline{B_\varepsilon}\right)$$

$$\underset{(7)}{\leq} \mu\left(\Phi(t)K + \overline{B_\varepsilon}\right)$$

$\Rightarrow$ Let $\varepsilon \to 0$. $\mu(K) \leq \mu(\Phi(t)K)$.

$\boxtimes$

Chap 2: Invariant Gibbs measures, Part 2.

Sec 2.1: Gibbs measures on $\mathbb{T}^2$.

We would like to construct

$$d\mu = Z^{-1} e^{-H(u)} du$$
$$= Z^{-1} e^{\mp \frac{1}{p}\int |u|^p} d\rho \quad \text{on } \mathbb{T}^2.$$

- ρ is supported on $H^s(\mathbb{T}^2) \setminus L^2(\mathbb{T}^2)$, $s<0$.

$\Rightarrow \int_{\mathbb{T}^2} |u|^p dx = \infty$ a.s.

$\Rightarrow$ We need to renormalize $\int |u|^p$

- In the following, we focus on the defocusing case.

(· Focusing case: Brydges–Slade '96.
We can not construct Gibbs meas for the focusing cubic NLS even if we consider the Wick ordered nonlinearity (with the Wick ordered L^2-cutoff.)

Sec 2.2: Hermite polynomicals, white noise functional, and Wick ordering

Hermite poly $H_k(x;\sigma)$ defined through the generating function

$$G(t,x;\sigma) = e^{tx - \frac{1}{2}\sigma t^2}$$

$$= \sum_{k=0}^{\infty} \frac{t^k}{k!} H_k(x;\sigma)$$

$$H_0(x;\sigma) = 1, \quad H_1(x;\sigma) = x$$

$$H_2(x;\sigma) = x^2 - \sigma, \quad H_3(x;\sigma) = x^3 - 3\sigma x,$$

$$H_4(x;\sigma) = x^4 - 6\sigma x^2 + 3\sigma^2$$

- White noise functional

$$W_{(\cdot)} : L^2(\mathbb{T}^2) \to L^2(\Omega)$$

by $W_f(\omega) = \langle f, w(\omega)\rangle_{L^2} = \sum_{n\in\mathbb{Z}^2} \widehat{f}(n)\, \overline{g_n(\omega)}$

where $w = \sum g_n(\omega) e^{in\cdot x}$ ← white noise on $\mathbb{T}^2$.

$\Rightarrow$ W_f is a Gaussian r.v with mean 0, var $\|f\|_{L^2}^2$

Moreover,

$$E[W_f \overline{W_h}] = \langle f, h\rangle_{L^2}.$$

For simplicity, we work in the real-valued setting. i.e. $g_{-n} = \overline{g_n}$, etc.

$\mathrm{Var}(g_n) = 1$

See Oh-Thomann '15 for $\mathbb{C}$-valued setting involving the (generalized) Laguerre polynomials.

Lemma 2.1 $f, h \in L^2(\mathbb{T}^2)$ with $\|f\|_{L^2} = \|h\|_{L^2} = 1$.

Then, we have

$$\mathbb{E}\left[H_k(W_f) H_m(W_h)\right] = \delta_{km} k! \left[\langle f, h\rangle_{L^2}\right]^k$$

Pf: First, recall

$$\int_\Omega e^{W_f(\omega)} dP = \prod_{n\in I^*} \frac{1}{\pi} \int_{\mathbb{C}} e^{2\,\mathrm{Re}(\hat{f}(n)\overline{g_n})} e^{-|g_n|^2} dg_n$$

$$\times \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{\hat{f}(0) g_0} e^{-\frac{1}{2} g_0^2} dg_0$$

$$= \prod_{n\in I^*} \frac{1}{\pi} \int_{\mathbb{R}} e^{2\,\mathrm{Re}\hat{f}(n)\,\mathrm{Re}\,g_n} e^{-(\mathrm{Re}\,g_n)^2} d\mathrm{Re}\,g_n \int_{\mathbb{R}} e^{2\,\mathrm{Im}\hat{f}(n)\,\mathrm{Im}\,g_n} e^{-(\mathrm{Im}\,g_n)^2} d\mathrm{Im}\,g_n$$

$$\times \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \cdots dg_0$$

$$= e^{\sum_{n\in I^*} |\hat{f}(n)|^2} e^{\frac{1}{2}(\hat{f}(0))^2} = e^{\frac{1}{2}\|f\|_{L^2}^2}.$$

I = index set = "$\mathbb{Z}^2/2$" $= \mathbb{Z}\times\mathbb{N} \cup \mathbb{N}\times\{0\} \cup \{(0,0)\}$

$I^* = I \setminus \{(0,0)\}$

$$\Rightarrow \int_\Omega G(t, W_f(\omega))\, G(s, W_h(\omega))\, dP(\omega)$$

$$= e^{-\frac{1}{2}(t^2+s^2)} \underbrace{\int_\Omega e^{W_{tf+sh}(\omega)}\, dP(\omega)}_{e^{\frac{1}{2}\|tf+sh\|^2_{L^2}}}$$

$$= e^{ts\langle f, h\rangle_{L^2}}$$

$$\Rightarrow e^{ts\langle f,h\rangle_{L^2}} = \sum_{k,m=0}^{\infty} \frac{t^k s^m}{k!\,m!} \int_\Omega H_k(W_f) H_m(W_h)\, dP$$

$$\|$$

$$\sum_{k=0}^{\infty} \frac{(ts)^k}{k!} \left(\langle f, h\rangle\right)^k.$$

$\boxtimes$

Under ρ, we have

$$u = \sum_{n\in\mathbb{Z}^2} \frac{g_n}{\langle n\rangle} e^{in\cdot x} \qquad \begin{array}{l} g_{-n} = \overline{g_n} \\ \mathrm{Var}(g_n) = 1 \end{array}$$

Given $N \in \mathbb{N}$, let $u_N := P_{\leq N} u$

Then, $u_N(x)$ is a mean 0 $\mathbb{R}$-valued Gaussian r.v.'s with variance

$$\sigma_N := \mathbb{E}[u_N^2(x)] = \sum_{|n|\leq N} \frac{1}{\langle n\rangle^2} \sim \log N.$$

Define the Wick ordered monomial

$$\boxed{:u_N^k(x): \;=\; H_k\,(u_N(x);\sigma_N)}$$

ex: Under ρ, we have

$$\int_{\mathbb{T}^2} u^2(\omega)\,dx = \infty \quad a.s.$$

but
$$\int_{\mathbb{T}^2} :u_N^2: \;=\; \int_{\mathbb{T}^2} u_N^2 - \sigma_N$$

$$= \sum_{|n|\le N} \frac{|g_n|^2-1}{\langle n\rangle^2}$$

and
$$\mathbb{E}\Big[\Big(\int_{\mathbb{T}^2} :u_N^2:\Big)^2\Big] = \mathbb{E}\big[(g_0^2-1)^2\big] + 2\,\mathbb{E}\sum_{\substack{n\in I^*\\ |n|\le N}} \frac{(|g_n|^2-1)^2}{\langle n\rangle^4}$$

$$\lesssim 1 + \sum_{|n|\le N} \frac{1}{\langle n\rangle^4} \lesssim 1 \quad \text{indep of } N$$

$\Rightarrow \int_{\mathbb{T}^2} :u_N^2:$ is a well-defined r.v.

and we can set

$$\int_{\mathbb{T}^2} :u^2: \;=\; \lim_{N\to\infty} \int_{\mathbb{T}^2} :u_N^2:$$

Let $$G_N(u) = \frac{1}{2m} \int_{\mathbb{T}^2} :(P_{\leq N} u)^{2m}: dx .$$

<u>Prop 2.2:</u> $\{G_N(u)\}$ is a Cauchy seq in $L^q(\rho)$ for any $q \geq 1$. Moreover, we have

$$\| G_M(u) - G_N(u) \|_{L^q(\rho)} \leq C_m (q-1)^m \frac{1}{N^{1/2}} .$$

for any $q \geq 1$, $M \geq N \geq 1$

This proposition allows us to define

$$G(u) = \frac{1}{2m} \int_{\mathbb{T}^2} : u^{2m} : dx$$

$$= \lim_{N \to \infty} G_N(u) \quad \text{in } L^q(\rho)$$

and the Wick ordered Hamiltonian

$$H_{Wick}(u) = \frac{1}{2} \int_{\mathbb{T}^2} |\nabla u|^2 + \frac{1}{2m} \int_{\mathbb{T}^2} : u^{2m} :$$

$e_n(y) = e^{in\cdot y}$

Pf: $(q=2)$

$$\eta_N(x)(\cdot) = \frac{1}{\sigma_N^{1/2}} \sum_{|n|\le N} \frac{\overline{e_n(x)}}{\langle n\rangle} e_n(\cdot)$$

$$\gamma_N(\cdot) = \sum_{|n|\le N} \frac{1}{\langle n\rangle^2} e_n(\cdot).$$

Note that

$$\|\eta_N(x)\|_{L^2(\mathbb{T}^2)} = 1 \quad \text{for each fixed } x\in\mathbb{T}^2$$

and

(9) $$\langle \eta_M(x), \eta_N(y)\rangle_{L^2(\mathbb{T}^2)} = \frac{1}{\sigma_M^{1/2}\sigma_N^{1/2}} \gamma_N(y-x).$$

$$u_N(x) = \sigma_N^{1/2} \frac{u_N(x)}{\sigma_N^{1/2}} = \sigma_N^{1/2} \overline{W_{\eta_N(x)}}$$

$$= \sigma_N^{1/2} W_{\eta_N(x)}.$$

$$\Rightarrow :u_N^{2m}(x): = H_{2m}(u_N(x); \sigma_N)$$

$$= \sigma_N^m H_{2m}\left(\frac{u_N(x)}{\sigma_N^{1/2}}; 1\right)$$

$$= \sigma_N^m H_{2m}\left(W_{\eta_N(x)}\right)$$

$$\Rightarrow (2m)^2 \| G_M(u) - G_N(u) \|^2_{L^2(P)}$$

$$= \int_{\mathbb{T}^2_x \times \mathbb{T}^2_y} \int_\Omega \sigma_M^{2m} H_{2m}(W_{\eta_M(x)}) H_{2m}(W_{\eta_M(y)})$$

$$- \sigma_M^m \sigma_N^m H_{2m}(W_{\eta_M(x)}) H_{2m}(W_{\eta_N(y)})$$

$$- \sigma_M^m \sigma_N^m H_{2m}(W_{\eta_N(x)}) H_{2m}(W_{\eta_M(y)})$$

$$+ \sigma_N^{2m} H_{2m}(W_{\eta_N(x)}) H_{2m}(W_{\eta_N(y)})$$

$$dP\, dx\, dy$$

$$\overset{\text{Lemma 2.1 and (9)}}{=} (2m)! \int_{\mathbb{T}^2_x \times \mathbb{T}^2_y} \left(\gamma_M(y-x)\right)^{2m} - \left(\gamma_N(y-x)\right)^{2m} dx\, dy$$

$$= (2m)! \int_{\mathbb{T}^2} \gamma_M(x)^{2m} - \gamma_N(x)^{2m}\, dx$$

$$\lesssim \int_{\mathbb{T}} |\gamma_M(x) - \gamma_N(x)| \left\{ |\gamma_M(x)|^{2m-1} + |\gamma_N(x)|^{2m-1} \right\} dx.$$

- $\| \gamma_M - \gamma_N \|_{L^2} = \left(\sum_{N < |n| \leq M} \frac{1}{\langle n \rangle^4} \right)^{\frac{1}{2}} \lesssim \frac{1}{N}$

- $\| \gamma_N^{2m-1} \|_{L^2} = \| \gamma_N \|^{2m-1}_{L^{4m-2}} \underset{\text{H-Y}}{\leq} \left(\sum_{|n| \leq N} \frac{1}{\langle n \rangle^{2 \cdot \frac{4m-2}{4m-3}}} \right)^{\frac{4m-3}{2}} \leq C_m < \infty$

indep of N.

$\square$

Sec 2.3: Hypercontractivity of the OU semigroup and Wiener chaos estimate

$$H = L^2(\mathbb{R}^d, \mu_d)$$

$\uparrow$ standard Gaussian meas on μ_d

$$L = \Delta - x \cdot \nabla$$

$=$ Hartree-Fock operator (Ornstein-Uhlenbeck op)

$\boxed{\text{Lem 2.3}}$ (Hypercontractivity of OU semigp, Nelson '65)

$p > 1$, $q \geq 1$. Then, we have

$$\| e^{tL} u \|_{L^q(\mathbb{R}^d, \mu_d)} \leq \| u \|_{L^p(\mathbb{R}^d, \mu_d)}$$

for $t \geq \frac{1}{2} \log\left(\frac{q-1}{p-1}\right)$.

Rmk: independent of dimension $d \geq 1$.

Define a homog Wiener chaos of order k

$$\mathbf{H}_k(x) = \prod_{j=1}^d H_{k_j}(x_j)$$

$$k = k_1 + \cdots + k_d$$

- $\mathcal{H}_k$ = closure of homog Wiener chaoses of order k under $L^2(\mathbb{R}^d, \mu_d)$

$\Rightarrow$ Ito-Wiener decomposition

$$L^2(\mathbb{R}^d, \mu_d) = \bigoplus_{k=0}^{\infty} \mathcal{H}_k.$$

$\Rightarrow$ $F \in \mathcal{H}_k$ is an eigenfunction of L with eigenvalue $-k$.

Cor 2.4: $F \in \mathcal{H}_k$. Then, for $q \geq 2$, we have

$$\|F\|_{L^q(\mathbb{R}^d, \mu_d)} \leq (q-1)^{k/2} \|F\|_{L^2(\mathbb{R}^d, \mu_d)}$$

($\Leftarrow$ Take $p=2$, $t = \frac{1}{2}\log(q-1)$ in Lemma 2.3.

Lemma 2.5

$$S_k(\omega) = \sum_{\Gamma(k,d)} C(n_1, \dots, n_k)\, g_{n_1}(\omega) \cdots g_{n_k}(\omega)$$

where $\Gamma(k,d) = \{(n_1, \dots, n_k) \in \mathbb{Z}^k : |n_j| \leq d\}$

$\Rightarrow$ Then, for $q \geq 2$,

$$\|S_k\|_{L^q(\Omega)} \leq \sqrt{k+1}\,(p-1)^{k/2} \|S_k\|_{L^2(\Omega)}$$

($\Leftarrow$ Cor 2.4 and $x^j = \sum_{m=0}^{[j/2]} C_{m,j}\, \sigma^m H_{j-2m}(x;\sigma)$

$\Rightarrow$ Prop 2.2 for general $q \geq 2$ follows from the $q=2$ case and Lemma 2.5.

(Note $G_M(u) - G_N(u) \in \bigcup_{k=0}^{2m} \mathcal{H}_k$.

Sec 2.4: Construction of the Gibbs measure on $\mathbb{T}^2$.

Construct

$$d\mu = Z^{-1} e^{-H_{Wick}(u)} du$$

$$= Z^{-1} e^{-\frac{1}{2m} \int_{\mathbb{T}^2} :u^{2m}: dx} d\rho$$

(φ_2^{2m} Euclidean quantum field theory)

Let $R_N(u) = e^{-G_N(u)} = e^{-\frac{1}{2m}\int :u_N^{2m}: dx}$

$\boxed{\text{Prop 2.6}}$: • $R_N(u) \in L^q(\rho)$ for any $q \geq 1$ with a unif bound in N.

• $R_N(u)$ converges to some $R(u)$ in $L^q(\rho)$.

Pf: $G_N(u)$ is NOT sign definite but
$-G_N(u)$ has a logarithmic upper bound:

(10)
$$-G_N(u) = -\frac{1}{2m}\int_{\mathbb{T}^2} \underbrace{H_{2m}(u_N;\sigma_N)}_{\substack{= \sigma_N^m H_{2m}\left(\frac{u_N}{\sigma_N}\right) \\ \geq -a_m}}\, dx$$
$$\leq b_m (\log N)^m.$$

Prop 2.2 yields

(11)
$$\rho\left(q\,|G_M(u) - G_N(u)| > \lambda\right) \leq C_m e^{-c_{m,q} N^{\frac{1}{2m}} \lambda^{\frac{1}{m}}}$$

for all $M \geq N \geq 1$, $q \geq 1$, $\lambda > 0$.

- $$\|R_N(u)\|^q_{L^q(\rho)} = \int_0^\infty \rho\left(e^{-qG_N(u)} > \alpha\right) d\alpha$$
$$\leq 1 + \int_1^\infty \rho\left(-qG_N(u) > \log\alpha\right) d\alpha$$

Hence, it suffices to show

(12)
$$\rho\left(-qG_N(u) > \log\alpha\right) \leq C\alpha^{-(1+\delta)}$$

for all $\alpha > 1$ and $N \in \mathbb{N}$

Given $\lambda = \log \alpha$,

choose $N_0 \in \mathbb{R}$ s.t. $\lambda = 2q\, bm (\log N_0)^m$.

Then, from (10), we have

$$\rho(-q\, G_N(u) > \lambda) = 0$$

for all $N < N_0$

- For $N \geq N_0$,

$$\rho(-q\, G_N(u) > \lambda)$$

$$\leq \rho\big(-q\, G_N(u) + q\, G_{N_0}(u) > \lambda - q\, bm (\log N_0)^m\big)$$

$$\leq \rho\big(-q\, G_N(u) + q\, G_{N_0}(u) > \tfrac{1}{2}\lambda\big)$$

$$\underset{(11)}{\leq} C_m \underbrace{e^{-c N_0^{1/2m} \lambda^{1/m}}}_{e^{-c\lambda^{1/m}} e^{c\lambda^{1/m}}}$$

$$\ll C e^{-(1+\delta)\lambda} \sim \alpha^{-(1+\delta)} \Rightarrow (12)$$

---

(11) $\Rightarrow$ $G_N(u)$ converges in measure

$\Rightarrow$ $R_N = e^{-G_N(u)}$ converges in measure.

By repeating the argument in Cor 1.4,

$R_N \to R = e^{-G(u)}$ in $L^q(\rho)$, $\forall q \geq 1$

$\square$

Rmk: The proof of Prop 2.6 is the so-called Nelson's estimate.

Prop 2.6 allows us to define the Gibbs measure

$$d\mu = Z^{-1} e^{-H_{Wick}(u)} du$$

$$= Z^{-1} e^{-\frac{1}{2m} \int_{\mathbb{T}^2} :u^{2m}:} d\rho.$$

Sec 2.5: A glimpse of probabilistic local Cauchy theory

- Wick ordered NLS:

$$i\partial_t u + \Delta u = :|u|^{2(m-1)}u:$$

- Wick ordered NLW:

$$-\partial_t^2 u + \Delta u = :u^{2m-1}:$$

- Wick ordered NLKG:

$$-\partial_t^2 u - u + \Delta u = :u^{2m-1}:$$

· Basic idea

- construct (local) solns a.s. with respect to ρ.

i.e. $u_0 = \sum_{n \in \mathbb{Z}^2} \frac{g_n}{\langle n \rangle} e^{in \cdot x}$

- gain of integrability of lin solns under randomization.

$\Rightarrow$ probabilistic Strichartz estimate

- Write $u = \underbrace{S(t) u_0}_{= z} + v$

z : random, rough (below $L^2(\mathbb{T}^2)$)

v : "deterministic", smoother (in $H^s(\mathbb{T}^2)$ for some $s > 0$)

Bourgain '96: cubic WNLS is a.s. LWP w.r.t. ρ.

$\Rightarrow$ a.s. GWP & invariance of μ

Case II in Chap 1.

- Also see Colliander-Oh '12.

Rmk: $s_{crit} = 1 - \frac{1}{m-1} \geq \frac{1}{2}$ for (super) quintic $m \geq 3$

$\Leftarrow$ gap is too big.

<u>NLW / NLKG:</u> gain of one derivative in the Duhamel operator.

$$:u_N^{2m-1}: = H_{2m-1}(z_N + v_N ; \sigma_N)$$

$$= \sum_{\ell=0}^{2m-1} \binom{2m-1}{\ell} v_N^{2m-1-\ell} H_\ell(z_N ; \sigma_N)$$

$$s_{crit} = \max\Big(\underbrace{1 - \frac{1}{m-1}}_{\text{from scaling}}, \underbrace{\frac{3}{4} - \frac{1}{2(m-1)}}_{\text{from Lorentz inv.}} \Big)$$

Prop 2.7: Probabilistic Strichartz estimates on Wiener homogeneous chaos.

$k \in \mathbb{N}$. Given $2 \leq q, r < \infty$ and $\alpha > 0$,

$$P\left(\| \langle \nabla \rangle^{-\alpha} H_k(z_N ; \sigma_N) \|_{L^q_T L^r_x} > \lambda \right)$$

$$\leq C \exp\left(-c \frac{\lambda^2}{T^{2/q}} \right)$$

for all $T, \lambda > 0$ and $N \in \mathbb{N}$.

<u>Rmk:</u> $k = 1 \Rightarrow$ "standard" prob Str. estimate.

$\alpha > 0$ is necessary since $z_N \in H^{-\varepsilon} \setminus L^2$ a.s.

Idea of Pf: Use the white noise functional and the orthogonality relation (Lemma 2.1).

→ $\boxed{\text{Theorem 2.8:}}$ (Oh - Thomann '15)

WNLW / WNLKG is a.s. LWP w.r.t. ρ.

($\Rightarrow$ a.s. GWP and invariance of ρ)

Rmk: Due to the lack of mass conservation, we need to consider WNLKG on $\mathbb{T}^2$. (or on a domain $M \subset \mathbb{R}^2$ with Neumann B.C.)

On $M \subset \mathbb{R}^2$ with Dirichlet B.C., we can treat WNLW as well.

Sec 2.6: Compactness argument. (Burq-Thomann-Tzvetkov '15

(Super) quintic WNLS on $\mathbb{T}^2$.

$s < 0$: support of the Gibbs meas μ.

$s_{crit} \geq \frac{1}{2}$: scaling critical regularity

$\Leftarrow$ The gap is too big!!

We do not know how to construct strong solutions, even probabilistically.

(FWNLS) $i\partial_t u^N + \Delta u^N = F_N(u^N)$

where $F^N(u) = P_{\le N}\left(:|P_{\le N} u|^{2(m-1)} P_{\le N} u: \right)$

$$= (-1)^{m+1}(m-1)!\, \sigma_N^{m-1} \cdot P_{\le N}\left\{ L_{m-1}^{(1)}\left(\frac{|u_N|^2}{\sigma_N}\right) u_N \right\}$$

generalized Laguerre poly.

- μ_N = invariant Gibbs meas for (FWNLS).

① Extend μ_N = prob. meas on initial data

to ν_N = prob meas on space-time functions

- (FWNLS) is globally well-posed.

$\Rightarrow \exists\, \Phi_N : u_0 \in \underset{s<0}{H^s} \longmapsto u^N \in C(\mathbb{R}; H^s)$

- Define a prob measure ν_N on $C(\mathbb{R}; H^s)$

by $\nu_N = \mu_N \circ \Phi_N^{-1}$

= induced probability meas under Φ_N

- By invariance of μ_N,

the law of $u^N(t) = \mu_N$ for any $t \in \mathbb{R}$.

② Show $\{\nu_N\}$ is tight (= compact)

$\underset{\text{Prokhorov}}{\Longrightarrow} \quad \nu_{N_j} \rightharpoonup \nu$

③ Skorokhod's Thm:

$\exists$ another prob space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P})$

$\widetilde{u^{N_j}}$, $C(\mathbb{R}, H^s)$-valued r.v's

u, "

s.t. $\mathcal{L}(\widetilde{u^{N_j}}) = \mathcal{L}(u^{N_j}) = \nu_N$

$\mathcal{L}(u) = \nu$

and $\widetilde{u^{N_j}}$ converges to u in $C(\mathbb{R}; H^s)$ almost surely w.r.t. $\tilde{P}$.

Rmk: $\mathcal{L}(\widetilde{u^{N_j}(t)}) = \mathcal{L}(u^{N_j}(t)) = \mu_N$

$\mathcal{L}(u(t)) = \mu$ (since $\mu_N \xrightarrow[\text{strongly}]{} \mu$)

$\boxed{\text{Theorem 2.9}}$ (Oh-Thomann '15)

$\exists$ a set Σ of full probability s.t.

$\forall \phi \in \Sigma$, $\exists$ soln $u \in C(\mathbb{R}; H^s)$ to (WNLS)

with $u|_{t=0} = \phi$.

Moreover, $\mathcal{L}(u(t)) = \mu$, $\forall t \in \mathbb{R}$.

Rmk: only a.s. global existence (NO uniqueness)

$\mathcal{D}'_{t,x}$ - valued r.v.

Idea: Let $X_j = i\partial_t \tilde{u}^{N_j} + \Delta \tilde{u}^{N_j} + F_{N_j}(\tilde{u}^{N_j})$.

Then, we have $\mathcal{L}(X_{N_j}) = \delta_0$

$\Rightarrow \mathcal{L}(X_\infty) = \delta_0$, i.e.

$u = \lim_{j\to\infty} \tilde{u}^{N_j}$ is a distri soln to (WNLS).

Need to show

$$:|u|^{2(m-1)}u: = \lim_{N\to\infty} F_N(u^N)$$

exists in $L^q(\rho; H^s(\mathbb{T}^2))$, $s<0$.

(analogous to Prop 2.2.

Sec 2.7: Further topics

① $\mathbb{T}^3$:

- Gibbs measure on $\mathbb{T}^3$ is renormalizable only for $p=4$

- Wick ordering is not enough.

  $\sigma_N \sim N$ in 3-d but need to replace it by $C_1 N + C_2 \log N$.

- stochastic quantization equation:

$$\partial_t u = \Delta u - u^3 + \infty \cdot u + \xi$$

"local well-posedness":

Hairer '14 (regularity structure)
Catellier-Chouk '14 (paracontrolled distribution)
Kupiainen '14 (renormalization group method)

- NLS / NLKG?

② On $\mathbb{R}$.

Take the periods $L \to \infty$. and construct the Gibbs meas on $\mathbb{R}$ as a diffusion process (Feynmann-Kac formula, Girsanov thm, etc.)

Oh-Quastel-Sosoe '13.

$\boxed{\text{Chap 3:}}$ Quasi-invariant Gaussian meas

$$d\rho_s = Z^{-1} e^{-\frac{1}{2}\|u\|_{H^s}^2} du$$

- ρ_s is supported on $H^{s-\frac{d}{2}-\varepsilon}(\mathbb{T}^d)$

$$\omega \longmapsto u = \sum_{n\in\mathbb{Z}^d} \frac{g_n}{\langle n\rangle^s} e^{in\cdot x}$$

Q: How is the Gaussian meas ρ_s transported under the nonlinear dynamics $\Phi(t): u_0 \mapsto u(t)$?

Is ρ_s quasi-invariant?

i.e. ρ_s and the pushforward meas $\Phi(t)_*\rho_s = \rho_s \circ \Phi(t)^{-1}$ are equivalent (= mutually abs. conti).

Or, can we prove that ρ_s and $\Phi(t)_*\rho_s$ are mutually singular?

Quasi-invariance

- Cameron-Martin '44:

$$S(t)\, u_0^\omega + \underbrace{(\tfrac{d}{2}+\varepsilon) \text{ smoother part}}_{}$$
indep of u_0

- Ramer '74

$$S(t)\, u_0^\omega + (d+\varepsilon) \text{ smoother part}$$
which <u>may depend</u> on u_0^ω

Crucial to establish a nonlinear smoothing.

- Fourth order cubic NLS (cubic biharmonic NLS)

(4NLS) $\quad i\partial_t u = \partial_x^4 u \pm |u|^2 u \quad$ on $\mathbb{T}$

There is no apparent nonlin. smoothing
$\Rightarrow$ We use normal form reduction.

Theorem 3.1: Oh - Tzvetkov '15

Let $s > \frac{3}{4}$. Then, μ_s is quasi-invariant under (4NLS).

Rmk: (4NLS) is globally well-posed in $L^2(\mathbb{T})$ (and ill-posed below L^2), so there is a gap $\frac{1}{2} < s < \frac{3}{4}$.

- The proof is based on a combination of global and local analysis.

See the earlier work (Tzvetkov '15) on the (generalized) BBM equation.

General strategy:

STEP 1: $\tilde{u}(t) = \mathcal{G}[u](t) := e^{2it\int |u|^2} u(t)$

$$\Rightarrow i\partial_t \tilde{u} = \partial_x^4 \tilde{u} + \left(|\tilde{u}|^2 - 2\int |u|^2 dx\right)\tilde{u}.$$

- $v(t) = S(-t)\,u(t)$ interaction representation.

(13) $$\Rightarrow \partial_t v_n = -i \sum_{\Gamma(n)} e^{-i\phi(\bar{n})} v_{n_1} \overline{v_{n_2}} v_{n_3} + i|v_n|^2 v_n$$

$$\Gamma(n) = \{ n = n_1 - n_2 + n_3,\ n_1, n_3 \neq n \}$$

$$\phi(\bar{n}) = n_1^4 - n_2^4 + n_3^4 - n^4$$
$$= (n_1 - n_2)(n_1 - n)\underbrace{\left(n_1^2 + n_2^2 + n_3^2 + n^2 + 2(n_1 + n_3)^2\right)}_{\text{rapid oscillation.}}$$

global probabilistic

① Show ρ_s is (quasi-) invariant under $\mathcal{G}$ and $S(-t)$. Also, under a composition of $T_1 \circ T_2$ where ρ_s is quasi-inv under each T_j.

$\Rightarrow$ Then, it suffices to show that ρ_s is quasi-invariant under (13).

STEP 2: Energy estimate.

local PDE analysis

$$\frac{d}{dt}\|v(t)\|^2_{H^s} = -2\,\mathrm{Re}\, i \sum_n \sum_{\Gamma(n)} e^{-i\phi(\bar{n})t} \langle n\rangle^{2s} v_{n_1}\overline{v_{n_2}} v_{n_3}\overline{v_{n_4}}$$

$$\overset{NF}{=} \boxed{2\,\mathrm{Re}\,\frac{d}{dt}\left[\sum_n \sum_{\Gamma(n)} \frac{e^{-i\phi(\bar{n})t}}{\phi(\bar{n})} \langle n\rangle^{2s} v_{n_1}\overline{v_{n_2}} v_{n_3}\overline{v_{n_4}}\right]} \; =: -R$$

$$-2\,\mathrm{Re}\sum_n \sum_{\Gamma(n)} \frac{e^{-i\phi(\bar{n})t}}{\phi(\bar{n})} \langle n\rangle^{2s} \underbrace{\partial_t\left(v_{n_1}\overline{v_{n_2}} v_{n_3}\overline{v_{n_4}}\right)}_{6\text{-lin}}$$

$\Rightarrow$ Defined a modified energy (as in the I-method)

$$E(v) = \|v\|^2_{H^s} + \boxed{R(v)}$$

Prop 3.2: Let $s > \frac{3}{4}$. $\exists\, \theta > 0$ s.t.

$$\left|\frac{d}{dt} E(P_{\leq N} v)\right| \leq C \|v\|^{4+\theta}_{L^2} \|v\|^{2-\theta}_{H^{s-\frac{1}{2}-\varepsilon}}$$

STEP 3: Construction of weighted Gaussian meas

$$d\mu_{s,N,r} = Z^{-1}_{s,N,r} \mathbb{1}_{\{\|v\|_{L^2} \leq r\}} e^{-\frac{1}{2}R(P_{\leq N} v)} d\rho_s$$

$$d\mu_{s,r} = Z^{-1}_{s,r} \mathbb{1}_{\{\|v\|_{L^2} \leq r\}} e^{-\frac{1}{2}R(v)} d\rho_s$$

- analogous to Chap 1.

STEP 4: change-of-variable formula.

$$d\mu_{s,N,r} = Z_{s,N,r}^{-1} \mathbb{1}_{\{\|v\|_{L^2} \le r\}} e^{-\frac{1}{2}E(P_{\le N} v)} dL_N \otimes d\rho_{s,N}^{\perp}$$

global

$$dL_N = \prod_{|n| \le N} d\hat{u}_n = \text{Leb meas on } \mathbb{C}^{2N+1}$$

$\boxed{\text{Prop 3.3.}}$ $s > 1/2$, $N \in \mathbb{N}$, $r > 0$. Then,

$$\mu_{s,N,r}(\Psi_N(t) A)$$

$$= Z^{-1} \int_{\Psi_N(t) A} \mathbb{1}_{\{\|v\|_{L^2} \le r\}} e^{-\frac{1}{2} R(P_{\le N} v)} d\rho_s(v)$$

$$= Z^{-1} \int_A \mathbb{1}_{\{\|v\|_{L^2} \le r\}} e^{-\frac{1}{2} E(P_{\le N} \Psi_N(t)(v))} dL_N \otimes d\rho_{s,N}^{\perp}.$$

$\Psi_N(t)$ = soln map to

$$\partial_t v_n = \mathbb{1}_{|n| \le N} \left\{ -i \sum_{\substack{\Gamma(n) \\ |n_j| \le N}} e^{-i\phi(\bar{n})t} v_{n_1} \bar{v}_{n_2} v_{n_3} + i |v_n|^2 v_n \right\}$$

- L_N is invariant under $\underbrace{P_{\le N} \Psi_N(t) P_{\le N}}_{\text{low freq dynamics}}$.

STEP 5: evolution of truncated measures

Lemma 3.4: Let $s > \frac{3}{4}$. Then, $\exists\, \beta \in [0,1)$ s.t. $\forall\, r\ \exists\, C > 0$ s.t.

$$\frac{d}{dt} \mu_{s,N,r}\left(\Psi_N(t)(A)\right) \le p^{\beta} \left\{\mu_{s,N,r}\left(\Psi_N(t)(A)\right)\right\}^{1-\frac{1}{p}}$$

for any $p \ge 2$, $N \in \mathbb{N}$, meas set $A \subset L^2(\mathbb{T})$

Pf: Idea: Reduce the problem to $t = 0$

$$\frac{d}{dt} \mu_{s,N,r}\left(\Psi_N(t)(A)\right)\Big|_{t=t_0}$$

$$= \frac{d}{dt} \mu_{s,N,r}\left(\Psi_N(t)\left(\Psi_N(t_0)(A)\right)\right)\Big|_{t=0}$$

$$\overset{\text{Prop 3.3}}{=} Z^{-1} \frac{d}{dt} \int_{\Psi_N(t_0)(A)} \mathbb{1}_{\{\|v\|_{L^2} \le r\}}\, e^{-\frac{1}{2}E(P_{\le N}\Psi_N(t)(v))}\, dL_N \otimes d\rho^{\perp}_{s,N}\Big|_{t=0}$$

$$= -\frac{1}{2} \int_{\Psi_N(t_0)(A)} \frac{d}{dt} E\left(P_{\le N}\Psi_N(t)(v)\right)\Big|_{t=0}\, d\mu_{s,N,r}$$

Now, apply Energy estimate (Prop 3.2) with Hölder's ineq.

$\boxtimes$

- Lemma 3.4 and Yudovich's argument

$\Rightarrow$ $\boxed{\text{Lemma 3.5}}$: Let $s > 3/4$. Given $t \in \mathbb{R}$, $r > 0$, and $\delta > 0$, $\exists$ $C = C(t, r, \delta) > 0$ s.t.

$$\mu_{s,N,r}(\Psi_N(t)(A)) \leq C\{\mu_{s,N,r}(A)\}^{1-\delta}$$

for any $N \in \mathbb{N}$ and measurable $A \subset L^2(\mathbb{T})$

STEP 6: Quasi-invariance of ρ_s.

- Lemma 3.5 and approximation argument

$\Rightarrow$ $\boxed{\text{Lemma 3.6}}$: Let $s > \frac{3}{4}$. Given $t \in \mathbb{R}$, $r > 0$, and $\delta > 0$, there exists $C = C(t, r, \delta) > 0$ s.t.

$$\mu_{s,r}(\Psi(t)(A)) \leq C\{\mu_{s,r}(A)\}^{1-\delta}$$

soln map for (13). on page 54.

Pf of Thm 3.1: Suppose $\rho_s(A) = 0$.

$\Rightarrow$ Then, $\rho_{s,r}(A) = 0$, where $d\rho_{s,r} = Z^{-1}\mathbb{1}_{\{\|v\|_{L^2} \leq r\}} d\rho_s$ for any $r > 0$.

$\Rightarrow$ $\mu_{s,r}(A) = 0$ b/c $\mu_{s,r} \ll \rho_{s,r}$

$\Rightarrow$ By Lemma 3.6, $\mu_{s,r}(\Psi(t)(A)) = 0$.

$\Rightarrow$ $\rho_{s,r}(\Psi(t)(A)) = 0$ b/c $\rho_{s,r} \ll \mu_{s,r}$.

$$\Rightarrow \rho_s(\Psi(t)A) \overset{\text{DCT}}{=} \lim_{r\to\infty} \rho_{s,r}(\Psi(t)A) = 0.$$

$\square$

Rmk: This argument allows us to obtain

$$\|u(t)\|_{H^\sigma} \lesssim t^{\alpha(\sigma)} \qquad \text{for} \quad \sigma < s - \tfrac{1}{2}$$

by adapting Bourgain's argument on p. 24